Supplementary Materials

Assessing tipping-point risk in carbon dioxide removal with a network-based framework

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S1 Introduction

This Supplementary Information collects the full technical derivations supporting the multistationarity and monostationarity results of the main paper. For each of the four Carbon Dioxide Removal (CDR) strategies analyzed there — biochar sequestration (BCS), ocean fertilization (OF), soil carbon sequestration (SCS), and wetland restoration (WR) — we give the complete proof of the corresponding proposition (Propositions 1–4 of the main text), worked out network by network. Section S2 gives precise definitions for chemical reaction networks with power-law kinetics, and Section S3 specifies the four CDR systems: their governing equations, their reaction network representations, and the network numbers that determine which multistationarity test applies to each. Section S4 then gathers the technical preliminaries used by the proofs but not needed for the main paper: the notation, the Deficiency One Algorithm (DOA) for power-law kinetic systems, the Multistationarity Algorithm (MSA), and the injectivity test of Wiuf and Feliu. Sections S5–S8 present the proofs for the BCS, OF, SCS and WR networks, respectively, and Section S9 examines how far those results depend on the storage parameterization. The numbering of species (A_1 land biota, A_2 atmosphere, A_3 ocean, A_4 total carbon stock, and A_i the CDR storage pool), the kinetic-order matrices, and the network classification by the Anderies ratio $R = (p_2 - p_1)/(q_2 - q_1)$ all follow the conventions established in the main text.

S2 Chemical Reaction Networks with Power-Law Kinetics

This appendix provides precise definitions of the chemical reaction network terms used throughout the manuscript, for the reader who wishes to consult them.

A *chemical reaction network* (CRN) is a triple $\mathscr{N} = (\mathscr{S}, \mathscr{C}, \mathscr{R})$ consisting of three finite sets. The *species* $\mathscr{S} = \{A_1, A_2, \ldots, A_m\}$ are the interacting components of the system; in the RNCDR framework, each species is a carbon pool. The *complexes* $\mathscr{C}$ are the formal nonnegative-integer combinations of species that appear on either side of a reaction arrow; for example, $A_1 + 2A_2$ and $2A_1 + A_2$ are each complexes. The *reactions* $\mathscr{R}$ are ordered pairs of complexes written with an arrow (e.g., $R_1 : A_1 + 2A_2 \to 2A_1 + A_2$), each representing a directional carbon transfer between pools.

The dynamics of a CRN are obtained by assigning a rate function to each reaction. In this study, every reaction R_i is assigned a *power-law rate* in the sense of Biochemical Systems Theory [7, 8, 9]

$$K_i(\mathbf{A}) = k_i \prod_{j=1}^{m} A_j^{F_{ij}},$$

where $k_i > 0$ is the reaction's *rate constant* and the *kinetic order* F_{ij} describes how strongly the rate of R_i depends on the concentration of species A_j. The matrix F whose entries are the kinetic orders is the *kinetic order matrix* of the network. A CRN together with such an assignment is called a *power-law kinetic system*, and its time evolution is governed by an ODE system in which each species' rate of change is the signed sum of the rate functions of the reactions producing and consuming it.

Several additional structural quantities are used in the multistationarity analysis of Section 4 of the main text and the network numbers reported in Section S3. The *reaction graph* of a CRN is the directed graph with complexes as nodes and reactions as edges. A *linkage class* is a connected component of this graph; the number of linkage classes is denoted ℓ. The *rank* s of the network is the dimension of the linear span of the reaction vectors (the differences between product and reactant complexes), which equals the dimension of the system's stoichiometric subspace. The *deficiency* of the network is the non-negative integer

$$\delta = n - \ell - s,$$

where n is the number of complexes [10]. Deficiency measures the degree of linear dependence among the reactions and determines which multistationarity-detection algorithms apply to the network: networks with $\delta = 1$ are amenable to the Deficiency One Algorithm, while networks with $\delta \geq 2$ require the Multistationarity Algorithm. For further background on these definitions and their use, see Supplementary Information.

S3 CDR System Details

This appendix collects the full ODE systems, CRN representations, and network numbers for each of the four CDR models analyzed in this study. The kinetic order matrices are presented alongside the CRN representations in Section 3.

ODE Systems

Biochar Sequestration.

$$\begin{aligned}
\dot{A}_1 &= k_1 A_1^{p_1} A_2^{q_1} - k_2 A_1^{p_2} A_2^{q_2} - k_{11,7} A_1 + k_{11,8} A_{11}, \\
\dot{A}_2 &= -k_1 A_1^{p_1} A_2^{q_1} + k_2 A_1^{p_2} A_2^{q_2} - a_m A_2 + a_m \beta A_3 + k_5 A_4^{e_{11}} A_{11}^{f_{11}}, \\
\dot{A}_3 &= a_m A_2 - a_m \beta A_3, \\
\dot{A}_4 &= k_{11,6} A_{11} - k_5 A_4^{e_{11}} A_{11}^{f_{11}}, \\
\dot{A}_{11} &= k_{11,7} A_1 - k_{11,6} A_{11} - k_{11,8} A_{11},
\end{aligned}$$

where $e_{11} \geq 1$, $f_{11} \leq 0$, and $e_{11} + f_{11} = 1$.

Ocean Fertilization.

$$\begin{aligned}
\dot{A}_1 &= k_1 A_1^{p_1} A_2^{q_1} - k_2 A_1^{p_2} A_2^{q_2}, \\
\dot{A}_2 &= -k_1 A_1^{p_1} A_2^{q_1} + k_2 A_1^{p_2} A_2^{q_2} - a_m A_2 + a_m \beta A_3 + k_5 A_4 - k_{12,7} A_2^{p_3} A_3^{q_3}, \\
\dot{A}_3 &= a_m A_2 - a_m \beta A_3 + k_{12,7} A_2^{p_3} A_3^{q_3} - k_{12,8} A_3^{p_4} + k_{12,9} A_3^{p_5} A_{12}^{q_5}, \\
\dot{A}_4 &= k_{12,6} A_{12} - k_5 A_4, \\
\dot{A}_{12} &= k_{12,8} A_3^{p_4} - k_{12,6} A_{12} - k_{12,9} A_3^{p_5} A_{12}^{q_5},
\end{aligned}$$

where $\lambda_{12} = 1$ and $\mu_{12} = 0$ so that the emission kinetics reduce to linear mass action: $K_5 = k_5 A_4$.

Soil Carbon Sequestration.

$$\begin{aligned}
\dot{A}_1 &= k_1 A_1^{p_1} A_2^{q_1} - k_2 A_1^{p_2} A_2^{q_2} - k_{13,7} A_1, \\
\dot{A}_2 &= -k_1 A_1^{p_1} A_2^{q_1} + k_2 A_1^{p_2} A_2^{q_2} - a_m A_2 + a_m \beta A_3 + k_5 A_4^{e_{13}} A_{13}^{f_{13}}, \\
\dot{A}_3 &= a_m A_2 - a_m \beta A_3, \\
\dot{A}_4 &= k_{13,6} A_{13} - k_5 A_4^{e_{13}} A_{13}^{f_{13}}, \\
\dot{A}_{13} &= k_{13,7} A_1 - k_{13,6} A_{13},
\end{aligned}$$

where $e_{13} \geq 1$, $f_{13} \leq 0$, and $e_{13} + f_{13} = 1$.

Wetland Restoration.

$$\begin{aligned}
\dot{A}_1 &= k_1 A_1^{p_1} A_2^{q_1} - k_2 A_1^{p_2} A_2^{q_2} - k_{14,9} A_1 + k_{14,10} A_{14}, \\
\dot{A}_2 &= -k_1 A_1^{p_1} A_2^{q_1} + k_2 A_1^{p_2} A_2^{q_2} - a_m A_2 + a_m \beta A_3 + k_5 A_4^{e_{14}} A_{14}^{f_{14}} + k_{14,7} A_2^{q_4} A_{14}^{p_4} - k_{14,8} A_2^{q_5} A_{14}^{p_5}, \\
\dot{A}_3 &= a_m A_2 - a_m \beta A_3, \\
\dot{A}_4 &= k_{14,6} A_{14} - k_5 A_4^{e_{14}} A_{14}^{f_{14}}, \\
\dot{A}_{14} &= -k_{14,7} A_2^{q_4} A_{14}^{p_4} - k_{14,6} A_{14} + k_{14,8} A_2^{q_5} A_{14}^{p_5} + k_{14,9} A_1 - k_{14,10} A_{14},
\end{aligned}$$

where $e_{14} \geq 1$, $f_{14} \leq 0$, and $e_{14} + f_{14} = 1$.

Complete CRN Representations

Biochar Sequestration.

$$\begin{aligned}
R_1 &: A_1 + 2A_2 \xrightarrow{k_1} 2A_1 + A_2 && \text{(atmosphere-to-land)} \\
R_2 &: 2A_1 + A_2 \xrightarrow{k_2} A_1 + 2A_2 && \text{(land-to-atmosphere)} \\
R_3 &: A_2 \xrightarrow{a_m} A_3 && \text{(atmosphere-to-ocean)} \\
R_4 &: A_3 \xrightarrow{a_m \beta} A_2 && \text{(ocean-to-atmosphere)} \\
R_5 &: A_4 \xrightarrow{k_5} A_2 && \text{(emission)} \\
R_{11,6} &: A_{11} \to A_4 && \text{(biochar to Total Carbon Stock)} \\
R_{11,7} &: A_1 \to A_{11} && \text{(land-to-biochar capture)} \\
R_{11,8} &: A_{11} \to A_1 && \text{(biochar-to-land transfer)}
\end{aligned}$$

Ocean Fertilization.

$$\begin{aligned}
R_1 &: 2A_2 + A_1 \xrightarrow{k_1} 2A_1 + A_2 && \text{(atmosphere-to-land)} \\
R_2 &: 2A_1 + A_2 \xrightarrow{k_2} 2A_2 + A_1 && \text{(land-to-atmosphere)} \\
R_3 &: A_2 \xrightarrow{a_m} A_3 && \text{(atmosphere-to-ocean)} \\
R_4 &: A_3 \xrightarrow{a_m \beta} A_2 && \text{(ocean-to-atmosphere)} \\
R_5 &: A_4 \xrightarrow{k_5} A_2 && \text{(emission)} \\
R_{12,6} &: A_{12} \to A_4 && \text{(OF to total carbon stock)} \\
R_{12,7} &: 2A_2 + A_3 \to A_2 + 2A_3 && \text{(atmosphere-to-ocean, photosynthesis)} \\
R_{12,8} &: 2A_3 \to A_3 + A_{12} && \text{(ocean-to-OF capture)} \\
R_{12,9} &: A_3 + A_{12} \to 2A_3 && \text{(OF-to-ocean release)}
\end{aligned}$$

Soil Carbon Sequestration.

$$\begin{aligned}
R_1 &: A_1 + 2A_2 \xrightarrow{k_1} 2A_1 + A_2 && \text{(atmosphere-to-land)} \\
R_2 &: 2A_1 + A_2 \xrightarrow{k_2} A_1 + 2A_2 && \text{(land-to-atmosphere)} \\
R_3 &: A_2 \xrightarrow{a_m} A_3 && \text{(atmosphere-to-ocean)} \\
R_4 &: A_3 \xrightarrow{a_m \beta} A_2 && \text{(ocean-to-atmosphere)} \\
R_5 &: A_4 \xrightarrow{k_5} A_2 && \text{(emission)} \\
R_{13,6} &: A_{13} \to A_4 && \text{(SCS to total carbon stock)} \\
R_{13,7} &: A_1 \to A_{13} && \text{(land-to-SCS capture)}
\end{aligned}$$

Wetland Restoration.

$$R_1 : A_1 + 2A_2 \xrightarrow{k_1} 2A_1 + A_2 \qquad \text{(atmosphere-to-land)}$$
$$R_2 : 2A_1 + A_2 \xrightarrow{k_2} A_1 + 2A_2 \qquad \text{(land-to-atmosphere)}$$
$$R_3 : A_2 \xrightarrow{a_m} A_3 \qquad \text{(atmosphere-to-ocean)}$$
$$R_4 : A_3 \xrightarrow{a_m\beta} A_2 \qquad \text{(ocean-to-atmosphere)}$$
$$R_5 : A_4 \xrightarrow{k_5} A_2 \qquad \text{(emission)}$$
$$R_{14,6} : A_{14} \to A_4 \qquad \text{(wetland to Total Carbon Stock)}$$
$$R_{14,7} : A_2 + 2A_{14} \to 2A_2 + A_{14} \qquad \text{(wetland respiration)}$$
$$R_{14,8} : 2A_2 + A_{14} \to A_2 + 2A_{14} \qquad \text{(wetland photosynthesis)}$$
$$R_{14,9} : A_1 \to A_{14} \qquad \text{(land-to-wetland capture)}$$
$$R_{14,10} : A_{14} \to A_1 \qquad \text{(wetland-to-land transfer)}$$

Network Numbers

Table S1 summarizes the network numbers for all four CDR systems, obtained using the CRN Toolbox [11]. All four systems share the same number of species ($m = 5$) and rank ($s = 4$). The BCS and SCS systems have deficiency 1, making them amenable to the Deficiency One Algorithm (DOA); the OF and WR systems have deficiency 2 and require the Multistationarity Algorithm (MSA) of Hernandez et al. [4]. The detailed computations and proofs are provided in the Supplementary Information.

Property	**BCS**	**OF**	**SCS**	**WR**
Species (m)	5	5	5	5
Complexes (n)	7	10	7	9
Reactions (r)	8	9	7	10
Linkage classes (ℓ)	2	4	2	3
Rank (s)	4	4	4	4
Deficiency (δ)	1	2	1	2

Table S1. Network numbers for the four CDR systems [11].

S4 Preliminaries

This section collects the definitions and algorithmic statements used in the proofs. It is intended to make the SI self-contained; readers familiar with chemical reaction network theory (CRNT) and the cited algorithms may skip to Section S5.

S4.1 Notation and conventions

To reiterate, a CRN is a triple $\mathscr{N} = (\mathscr{S}, \mathscr{C}, \mathscr{R})$ consisting of a finite set of species $\mathscr{S}$, a finite set of complexes $\mathscr{C}$ (nonnegative-integer combinations of species), and a finite set of reactions $\mathscr{R} \subseteq \mathscr{C} \times \mathscr{C}$. A reaction is written $y \to y'$, where y is the *reactant complex* and y' the *product complex*; $\rho(\mathscr{R})$ denotes the set of reactant complexes. The *stoichiometric subspace* S is the linear span of the reaction vectors $\{y' - y\}$, and its dimension is the *rank* s of the network. Two states $x, x^* \in \mathbb{R}^{\mathscr{S}}_{>0}$ are *stoichiometrically compatible* if $x^* - x \in S$.

Each reaction is endowed with a power-law rate

$$K_i(\mathbf{A}) = k_i \prod_j A_j^{F_{ij}}, \qquad k_i > 0, \tag{S1}$$

where the kinetic order F_{ij} records the influence of species A_j on reaction R_i and F is the kinetic order matrix. A power-law system is *reactant-determined* (*PL-RDK*) if reactions sharing the same reactant complex have identical kinetic-order rows; equivalently, the kinetic orders are organized in the T-matrix, whose columns are indexed by the reactant complexes (Definition A.14 of [3]). We write $T_{\cdot i}$ for the column of T associated with reactant complex i, and "$\cdot$" for the standard scalar product on $\mathbb{R}^{\mathscr{S}}$. Throughout,

$\mu \in \mathbb{R}^{\mathscr{S}}$ is the logarithmic displacement $\mu = \log x^* - \log x$ between two candidate equilibria; a nonzero μ that is sign-compatible with S is called a *signature*.

A vector $\mu \in \mathbb{R}^{\mathscr{S}}$ is *sign-compatible* with S if there exists $\sigma \in S$ with $\mathrm{sign}(\mu_s) = \mathrm{sign}(\sigma_s)$ for all $s \in \mathscr{S}$. The existence of a signature is, for the network classes considered here, equivalent to the capacity for multiple positive steady states (*multistationarity*); its non-existence certifies *monostationarity*.

The *deficiency* of the network is $\delta = n - \ell - s$, where $n = |\mathscr{C}|$ and ℓ is the number of linkage classes. The BCS and SCS networks have $\delta = 1$ (handled by the DOA), while the OF and WR networks have $\delta = 2$ (handled by the MSA). The four networks, their reactions, and their kinetic-order matrices are as specified in the main paper; we restate the T-matrices locally in each proof.

S4.2 The Deficiency One Algorithm for power-law kinetic systems

For a regular deficiency-one PL-RDK system, the DOA of Fortun et al. [3] translates the (nonlinear) multistationarity question into the solvability of a linear system of equalities and inequalities. It rests on the following equivalence (Theorem 3.1 of [3]): a regular deficiency-one PL-RDK system $(\mathscr{N}, K)$ admits two distinct, stoichiometrically compatible positive equilibria for some choice of rate constants if and only if there exist a confluence vector h and an upper–middle–lower (UML) partition $\{U, M, L\}$ of the reactant complexes such that the polyhedron induced by h and $\{U, M, L\}$ contains a nonzero vector μ that is sign-compatible with S.

A *confluence vector* $h \in \mathbb{R}^{\mathscr{C}}$ satisfies $\sum_{i \in \mathscr{C}} h_i i = 0$, $\sum_{i \in \mathscr{L}} h_i = 0$ for each linkage class $\mathscr{L}$, and $\sum_{i \in \mathscr{A}} h_i > 0$ for each terminal strong linkage class $\mathscr{A}$ that is not itself a linkage class. An *UML partition* divides the reactant complexes into parts U (upper), M (middle), and L (lower) subject to the structural conditions of Definition 2.5 of [3]. The induced polyhedron $\mathscr{P} = \mathscr{P}^1 \cap \mathscr{P}^2 \cap \mathscr{P}^3$ collects: equalities $T_{\cdot i} \cdot \mu = T_{\cdot j} \cdot \mu$ for $i, j \in M$ ($\mathscr{P}^1$); strict inequalities $T_{\cdot j} \cdot \mu > T_{\cdot i} \cdot \mu$ whenever complex j lies above complex i ($\mathscr{P}^2$); and the sign relations on cut pairs in U and L governed by $h(\mathscr{W}(i))$ ($\mathscr{P}^3$), where $h(\mathscr{W}(i))$ is the sum of the components of h over the linkage-class fragment containing i.

In practice the algorithm proceeds as: (i) find a confluence vector h; (ii) choose a UML partition; (iii) collect the equalities among complexes of M; (iv) collect the inequalities between parts; (v) collect the cut-pair relations between U and L; (vi) assemble the linear system; and (vii) test for a nonzero solution μ that is sign-compatible with S. If such a μ exists, the system is multistationary; otherwise one returns to (ii) and tries another partition.

S4.3 The Multistationarity Algorithm

For networks of deficiency $\delta \geq 2$, the MSA of Hernandez et al. [4] extends the DOA. One first selects an *orientation* $\mathscr{O} \subseteq \mathscr{R}$ (one reaction from each reversible pair, all irreversible reactions) and the associated linear map $L_{\mathscr{O}}(\alpha) = \sum_{y \to y' \in \mathscr{O}} \alpha_{y \to y'} (y' - y)$. The relation $y \to y' \sim y \to y''$ (when their $\ker L_{\mathscr{O}}$ coordinates are proportional) partitions $\mathscr{O}$ into *equivalence classes* P_i and, after grouping reversible pairs, into *fundamental classes* C_i. A representative set $W = \{y_i \to y_i'\}$ is chosen and a basis $\{b^j\}$ of $\ker^{\perp} L_{\mathscr{O}} \cap \Gamma_W$ is computed, where Γ_W is the set of vectors supported on W.

For a confluence-like pair $g_W, h_W \in \mathbb{R}^{\mathscr{O}} \cap \Gamma_W$, each reaction is *shelved* upper/middle/lower according to whether $e^{T_{\cdot y} \cdot \mu}$ exceeds, equals, or falls below the ratio $\rho_{y_i \to y_i'} = h_{y_i \to y_i'} / g_{y_i \to y_i'}$, with $M_i = \ln \rho_{y_i \to y_i'}$ (taken large and negative when $\rho \leq 0$). The shelving assignments, together with the sign conditions on g_W and h_W and the linearity test (the basis graph of $\ker^{\perp} L_{\mathscr{O}} \cap \Gamma_W$ must be a forest), generate a linear system in μ. By Theorem 4 of [4], the network has the capacity for multistationarity if and only if this system admits a signature μ that is sign-compatible with S. The algorithm is applied through the fourteen steps of [4]; we follow that numbering in the OF and WR proofs.

S4.4 The injectivity test of Wiuf and Feliu

To certify monostationarity we use the determinant-based injectivity test of Wiuf and Feliu [2, 1]. For a power-law system with stoichiometric matrix N and kinetic-order matrix F, form

$$M = N \,\mathrm{diag}(z)\, F \,\mathrm{diag}(k), \tag{S2}$$

with symbolic vectors $z = (z_1, \ldots, z_r)$ and $k = (k_1, \ldots, k_r)$. Let $\{\omega^1, \ldots, \omega^d\}$ be a basis of the left kernel of N (the conservation laws), and form M^* by replacing d suitably chosen rows of M by these basis vectors. By Theorem 2 of [1], the network is *injective* — and hence admits at most one positive steady state in each stoichiometric class — if and only if $\det(M^*)$ is a nonzero homogeneous polynomial in z, k whose coefficients are either all positive or all negative. If $\det(M^*)$ contains two terms of opposite sign (or is identically zero), the network is *non-injective*; non-injectivity is a necessary, but not sufficient, condition for multistationarity. All determinants below are computed with the Maple script of [1].

S5 Proof of Proposition 1 (Biochar Sequestration)

The BCS network has deficiency 1, so the DOA of Section S4.2 applies. We first set up the confluence vector, the T-matrix, and the UML partition used throughout, then prove the four parts of the proposition.

S5.1 Setup: confluence vector, T-matrix, and UML partition

Confluence vector. Ordering the complexes as A_1+2A_2, $2A_1+A_2$, A_1, A_2, A_3, A_4, A_{11}, a confluence vector of the BCS network is

$$h = \begin{bmatrix} h_{A_1+2A_2} \\ h_{2A_1+A_2} \\ h_{A_1} \\ h_{A_2} \\ h_{A_3} \\ h_{A_4} \\ h_{A_{11}} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{S3}$$

T-matrix. With the species ordered $A_1, A_2, A_3, A_4, A_{11}$, the T-matrix of the BCS system is

$$T = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_{11} \end{array} \begin{array}{c} \begin{array}{ccccccc} A_1{+}2A_2 & 2A_1{+}A_2 & A_1 & A_2 & A_3 & A_4 & A_{11} \end{array} \\ \begin{bmatrix} p_1 & p_2 & 1 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & f_{11} & 1 \end{bmatrix} \end{array}. \tag{S4}$$

UML partition. Among the nine admissible UML partitions of the BCS reactant complexes (Table S2), the proofs use **Partition 3**:

$$U = \{A_1 + 2A_2,\ 2A_1 + A_2\}, \quad M = \{A_1, A_4, A_{11}\}, \quad L = \{A_2, A_3\}.$$

Partition	Upper	Middle	Lower
1	$\{A_1{+}2A_2, 2A_1{+}A_2, A_2, A_3\}$	$\{A_1, A_{11}, A_4\}$	$\emptyset$
2	$\emptyset$	$\{A_1, A_{11}, A_4\}$	$\{A_1{+}2A_2, 2A_1{+}A_2, A_2, A_3\}$
3	$\{A_1{+}2A_2, 2A_1{+}A_2\}$	$\{A_1, A_{11}, A_4\}$	$\{A_2, A_3\}$
4	$\{A_2, A_3\}$	$\{A_1, A_{11}, A_4\}$	$\{A_1{+}2A_2, 2A_1{+}A_2\}$
5	$\{A_1{+}2A_2, 2A_1{+}A_2\}$	$\{A_1, A_{11}, A_4, A_2, A_3\}$	$\emptyset$
6	$\emptyset$	$\{A_1, A_{11}, A_4, A_2, A_3\}$	$\{A_1{+}2A_2, 2A_1{+}A_2\}$
7	$\{A_2, A_3\}$	$\{A_1, A_{11}, A_4, A_1{+}2A_2, 2A_1{+}A_2\}$	$\emptyset$
8	$\emptyset$	$\{A_1, A_{11}, A_4, A_1{+}2A_2, 2A_1{+}A_2\}$	$\{A_2, A_3\}$
9	$\emptyset$	$\mathscr{C}$	$\emptyset$

Table S2. Upper–middle–lower partitions of the reactant complexes of the BCS network.

Linear system from Partition 3. Writing $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_{11})^\mathsf{T}$ for the displacements associated with $(A_1, A_2, A_3, A_4, A_{11})$, the equalities among the complexes of M give $\mu_1 = e_{11}\mu_4 + f_{11}\mu_{11} = \mu_{11}$, which (using $e_{11} + f_{11} = 1$) simplifies to $\mu_1 = \mu_4 = \mu_{11}$. Collecting the inequalities between parts (U above M above L) and the cut-pair relations between U and L yields the summarized system

$$\begin{aligned} &\mu_1 = \mu_4 = \mu_{11}, \\ &p_2\mu_1 + q_2\mu_2 > p_1\mu_1 + q_1\mu_2 > \mu_1 > \mu_3 \geq \mu_2, \end{aligned} \tag{S5}$$

which is equivalent to

$$-\mu_1 \frac{p_2 - p_1}{q_2 - q_1} > \mu_2 > \mu_1 \frac{1 - p_1}{q_1}, \qquad q_1 \neq q_2,\ q_1 \neq 0. \tag{S6}$$

The stoichiometric subspace of the BCS system is

$$S = \left\{ \begin{bmatrix} a_1 - a_4 \\ -a_1 - a_2 + a_3 \\ a_2 \\ -a_3 \\ a_4 \end{bmatrix} \middle| \, a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}, \tag{S7}$$

and the candidate displacement has the form $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)^\mathsf{T}$.

Proposition 1 (Biochar Sequestration)**.** *The following hold for the BCS system:*

(i) ***Positive class.*** *If $p_2 > p_1$ and $q_2 > q_1$, or $p_1 > p_2$ and $q_1 > q_2$, then the system is multistationary.*

(ii) ***Negative class.*** *If $p_1 < 0$, $p_2 > 0$, $q_1 > 0$, $q_2 < 0$, $e_{11} \geq 1$, and $f_{11} \leq 0$, then the system is monostationary.*

(iii) ***P-null class.*** *The system is multistationary.*

(iv) ***Q-null class.*** *The system is multistationary.*

S5.2 Proof of part (i): the positive class is multistationary

Let $R_p = (p_2 - p_1)/(q_2 - q_1) > 0$, i.e. $p_2 > p_1$ and $q_2 > q_1$. From the setup, the displacement takes the form $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)^\mathsf{T}$ subject to (S6).

Suppose $\mu_1 > 0$. For sign-compatibility of μ with S in (S7) we then need $a_1 > a_4 > 0$ and $a_3 < 0$. Because $R_p > 0$, the left-hand side of (S6) is negative, so $\mu_2 < 0$; hence the second coordinate $-a_1 - a_2 + a_3$ of S must be negative, which forces $a_2 > 0$. Consequently the third coordinate $a_2 > 0$ is sign-compatible with $\mu_3 > 0$. Thus a nonzero μ sign-compatible with some $\sigma \in S$ exists, and the positive-class BCS system is multistationary. The reversed case $p_1 > p_2$, $q_1 > q_2$ is symmetric. □

S5.3 Proof of part (ii): the negative class is monostationary

Now let $R_p = (p_2 - p_1)/(q_2 - q_1) < 0$, realized by $p_1 < 0$, $p_2 > 0$, $q_1 > 0$, $q_2 < 0$. We certify monostationarity through the injectivity test of Section S4.4. The Maple script of [1] gives the determinant

$$\begin{aligned} \det(M^*) = & -e_{11}p_1k_1k_2k_4k_5z_1z_3z_5z_6 - e_{11}p_1k_1k_2k_4k_5z_1z_3z_5z_8 \\ & -e_{11}p_1k_1k_3k_4k_5z_1z_4z_5z_6 - e_{11}p_1k_1k_3k_4k_5z_1z_4z_5z_8 \\ & +e_{11}p_2k_1k_2k_4k_5z_2z_3z_5z_6 + e_{11}p_2k_1k_2k_4k_5z_2z_3z_5z_8 \\ & +e_{11}p_2k_1k_3k_4k_5z_2z_4z_5z_6 + e_{11}p_2k_1k_3k_4k_5z_2z_4z_5z_8 \\ & +e_{11}q_1k_1k_2k_3k_4z_1z_4z_5z_7 + e_{11}q_1k_2k_3k_4k_5z_1z_4z_5z_6 \\ & +e_{11}q_1k_2k_3k_4k_5z_1z_4z_5z_8 - e_{11}q_2k_1k_2k_3k_4z_2z_4z_5z_7 \\ & -e_{11}q_2k_2k_3k_4k_5z_2z_4z_5z_6 - e_{11}q_2k_2k_3k_4k_5z_2z_4z_5z_8 \\ & -f_{11}q_1k_1k_2k_3k_5z_1z_4z_5z_7 + f_{11}q_2k_1k_2k_3k_5z_2z_4z_5z_7 \\ & +e_{11}k_1k_2k_4k_5z_3z_5z_6z_7 + e_{11}k_1k_3k_4k_5z_4z_5z_6z_7 \\ & +q_1k_1k_2k_3k_5z_1z_4z_6z_7 - q_2k_1k_2k_3k_5z_2z_4z_6z_7. \end{aligned} \tag{S8}$$

Since $e_{11} \geq 1$ and $f_{11} \leq 0$, the sign conditions $p_1 < 0$, $p_2 > 0$, $q_1 > 0$, $q_2 < 0$ make every term of (S8) positive. By the test of Section S4.4, the BCS network is injective, and therefore monostationary. □

S5.4 Proof of parts (iii) and (iv): the P-null and Q-null classes are multistationary

(iii) P-null class ($R_p = 0$). Here $p_2 = p_1$ and $q_2 \neq q_1$. Partition 3 again yields a displacement $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)^\mathsf{T}$ subject to (S6). With $R_p = 0$ the left-hand side of (S6) vanishes, so $0 > \mu_2 > \mu_1(1 - p_1)/q_1$ with $q_1 \neq 0$; hence $\mu_2 < 0$, which makes $-a_1 - a_2 + a_3 < 0$. Taking $\mu_1 > 0$ forces $a_1 > a_4 > 0$ and $a_3 < 0$, whence $a_2 > 0$ and so $\mu_3 > 0$. A nonzero μ sign-compatible with S therefore exists, and the P-null BCS system is multistationary.

(iv) Q-null class ($R_q = 0$). Here $q_2 = q_1$ and $p_2 \neq p_1$. Using Partition 3, the analogue of (S6) reads

$$-\mu_2 \frac{q_2 - q_1}{p_2 - p_1} > \mu_1 > \mu_2 \frac{q_1}{1 - p_1}, \qquad p_1 \neq p_2,\ p_1 \neq 0. \tag{S9}$$

With $R_q = (q_2 - q_1)/(p_2 - p_1) = 0$ the left-hand side vanishes, giving $0 > \mu_1 > \mu_2 q_1/(1 - p_1)$ with $p_1 \neq 0$, so $\mu_1 < 0$. Sign-compatibility then yields $a_1 < a_4 < 0$ and $a_3 > 0$; taking $\mu_3 < 0$ gives $a_2 < 0$, hence $-a_1 - a_2 + a_3 > 0$ and $\mu_2 > 0$. A nonzero μ sign-compatible with S exists, so the Q-null BCS system is multistationary. □

S6 Proof of Proposition 2 (Ocean Fertilization)

The OF network has deficiency 2, so the MSA of Section S4.3 applies. We first run the algorithm to obtain the master solution μ, then specialize it to prove the four parts.

S6.1 Setup: MSA preliminaries for the OF network

Step 1 — Orientation. Choose $\mathscr{O} = \{R_1, R_3, R_5, R_{12,6}, R_{12,7}, R_{12,8}\}$, with map

$$L_{\mathscr{O}} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_{12} \end{array} \begin{array}{c} \begin{array}{cccccc} R_1 & R_3 & R_5 & R_{12,6} & R_{12,7} & R_{12,8} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \end{array}.$$

Step 2 — Equivalence and fundamental classes. A basis of $\ker L_{\mathscr{O}}$ is $v_1 = (0,-1,0,0,1,0)^{\mathsf{T}}$ and $v_2 = (0,1,1,1,0,1)^{\mathsf{T}}$, giving the classes

$$\begin{array}{ll} P_0 = \{R_1\}, & C_0 = \{R_1, R_2\}; \\ P_1 = \{R_3\}, & C_1 = \{R_3, R_4\}; \\ P_2 = \{R_5, R_{12,6}, R_{12,8}\}, & C_2 = \{R_5, R_{12,6}, R_{12,8}, R_{12,9}\}; \\ P_3 = \{R_{12,7}\}, & C_3 = \{R_{12,7}\}. \end{array}$$

The reactions of P_0 are reversible, and P_2 admits $\alpha = 1 > 0$ with $v^l_{R_5} = \alpha v^l_{R_{12,6}} = \alpha v^l_{R_{12,8}}$, so the algorithm proceeds.

Steps 3–7 — Colinkage sets, representatives, linearity. The terminal strong colinkage classes are $\{2A_2{+}A_1, 2A_1{+}A_2\}$, $\{A_2, A_3\}$, $\{A_2\}$, $\{2A_3, A_3{+}A_{12}\}$, and $\{A_2{+}2A_3\}$. Pick the representatives $W = \{R_3, R_5, R_{12,7}\}$ (R_3 for P_1, R_5 for P_2, $R_{12,7}$ for P_3); no realignment is needed. Restricting $\ker^{\perp} L_{\mathscr{O}} \cap \Gamma_W$ to the rows of W gives the single basis vector $a^1 = (1,-1,1)^{\mathsf{T}}$. Its basis graph is the star on $\{R_3, R_5, R_{12,7}, a^1\}$, which is acyclic; hence the basis is a forest and the resulting inequality system is linear.

Step 8 — Sign patterns. Taking $g = (0,1,3,3,2,3)^{\mathsf{T}}$ and $h = (0,1,2,2,1,2)^{\mathsf{T}}$ in $\ker L_{\mathscr{O}}$ gives $g_W = (1,3,2)^{\mathsf{T}} > 0$ and $h_W = (1,2,1)^{\mathsf{T}} > 0$, so both sign patterns are positive on every P_i ($i = 1,2,3$), including the nonreversible class; the pair is compatible.

Step 9–12 — Shelving and the T-matrix. The T-matrix of the OF system is

$$T = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_{12} \end{array} \begin{array}{c} \begin{array}{ccccccccc} 2A_2{+}A_1 & 2A_1{+}A_2 & A_2 & A_3 & A_4 & A_{12} & 2A_2{+}A_3 & 2A_3 & A_3{+}A_{12} \end{array} \\ \begin{bmatrix} p_1 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & 1 & 0 & 0 & 0 & p_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_3 & p_4 & p_5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_5 \end{bmatrix} \end{array}. \tag{S10}$$

The nondegenerate shelving is $\mathscr{L}_1 = \{R_3, R_4\}$; $\mathscr{M}_2 = \{R_5, R_{12,6}\}$, $\mathscr{L}_2 = \{R_{12,8}, R_{12,9}\}$; $\mathscr{M}_3 = \{R_{12,7}\}$. Letting each M_i be a large negative number, the shelving equalities and inequalities are

$$\begin{aligned}
&\mu_{A_2} < M_1, \quad \mu_{A_3} < M_1,\\
&\mu_{A_4} = M_2, \quad \mu_{A_{12}} = M_2, \quad p_4\mu_{A_3} < M_2, \quad p_5\mu_{A_3} + q_5\mu_{A_{12}} < M_2,\\
&p_3\mu_{A_2} + q_3\mu_{A_3} = M_3,\\
&p_1\mu_{A_1} + q_1\mu_{A_2} = p_2\mu_{A_1} + q_2\mu_{A_2} \quad \text{(from } P_0\text{)},
\end{aligned} \tag{S11}$$

together with the lower-shelf relations $\mu_{A_2} > \mu_{A_3}$ (from C_1) and $p_4\mu_{A_3} > p_5\mu_{A_3} + q_5\mu_{A_{12}}$ (from C_2). *The quantity M_3 is, by the third equality of* (S11)*, defined by*

$$\boxed{M_3 := p_3\mu_{A_2} + q_3\mu_{A_3}\,,} \tag{S12}$$

the shelf level of the photosynthesis reaction $R_{12,7}$; it is taken large and negative.

Steps 13–14 — Master solution. From $a^1 = (1, -1, 1)$ one has $R^1_+ = \{y_1 \to y_1', y_3 \to y_3'\}$, $R^1_- = \{y_2 \to y_2'\}$ and, since all $g_{y\to y'} > 0$, the degenerate set is empty ($D = \emptyset$); the nonsegregated cases are $M_1 < M_2 < M_3$, $M_1 = M_2 = M_3$, and $M_1 > M_2 > M_3$. Solving (S11) gives the master displacement

$$\mu = \begin{bmatrix} \mu_{A_1}\\ \mu_{A_2}\\ \mu_{A_3}\\ \mu_{A_4}\\ \mu_{A_{12}} \end{bmatrix} = \begin{bmatrix} \dfrac{q_2 - q_1}{p_2 - p_1}(-\mu_{A_2})\\ \mu_{A_2}\\ \dfrac{M_3 - p_3\mu_{A_2}}{q_3}\\ M_2\\ M_2 \end{bmatrix}, \qquad p_1 \neq p_2,\ q_3 \neq 0, \tag{S13}$$

with $\mu_{A_2}, \mu_{A_3}, \mu_{A_4}, \mu_{A_{12}} < 0$. The stoichiometric subspace is

$$S = \left\{ \begin{bmatrix} w\\ -w - x + y\\ x\\ -y + z\\ -z \end{bmatrix} \;\middle|\; w, x, y, z \in \mathbb{R} \right\}. \tag{S14}$$

From the signs of μ one reads $z > 0$, $y > 0$, $x < 0$, so sign-compatibility requires $w > 0$; that is, $\mu_{A_1} > 0$. By (S13) this holds precisely when the ratio $(q_2 - q_1)/(p_2 - p_1) > 0$ (since $-\mu_{A_2} > 0$). This is the master criterion used below.

Proposition 2 (Ocean Fertilization)**.** *The following hold for the OF system:*

- *(i)* ***Positive class.*** *If $q_1 > q_2 > 0$ and $p_1 > p_2 > 0$ (or $q_2 > q_1 > 0$ and $p_2 > p_1 > 0$), together with $p_3, q_3, p_4, p_5, q_5 > 0$ and $p_3\mu_{A_2} > M_3$, then the system is multistationary.*
- *(ii)* ***Negative class.*** *If $q_2 > q_1 > 0$ and $p_1 > 0 > p_2$, together with $p_3, q_3, p_5, q_5 > 0$, $p_4 < 0$, and $p_3\mu_{A_2} > M_3$, then the system is monostationary.*
- *(iii)* ***Q-null class.*** *If $q_2 = q_1$ and $p_1 > 0 > p_2$, together with $p_3, q_3, p_5, q_5 > 0$, $p_4 < 0$, and $p_3\mu_{A_2} > M_3$, then the system is monostationary.*
- *(iv)* ***P-null class.*** *If $p_2 = p_1$, $q_2 \neq q_1$, and $q_3 > 0 > p_3$, then the system is monostationary.*

S6.2 Proof of part (i): the positive class is multistationary

Take μ as in (S13) with $\mu_{A_3} < \mu_{A_2} < M_1 < 0$ and $\mu_{A_4} = \mu_{A_{12}} = M_2 < 0$, and $M_3 < 0$, $p_1 \neq p_2$, $q_3 > 0$.

Sign of μ_{A_3} and the condition $p_3\mu_{A_2} > M_3$. Suppose $q_3 > 0$, $p_3 > 0$, and $p_3\mu_{A_2} > M_3$. Since $p_3 > 0$ and $\mu_{A_2} < 0$, we have $p_3\mu_{A_2} < 0$; and since $M_3 < 0$ with $p_3\mu_{A_2} > M_3$, it follows that $M_3 - p_3\mu_{A_2} < 0$. Hence $\mu_{A_3} = (M_3 - p_3\mu_{A_2})/q_3 < 0$, as required by the shelving inequalities.

From the master criterion, $\mu_{A_1} > 0$ requires $(q_2 - q_1)/(p_2 - p_1) > 0$. Under $q_1 > q_2 > 0$ and $p_1 > p_2 > 0$ (or $q_2 > q_1 > 0$ and $p_2 > p_1 > 0$), the ratio $R_Q = (q_2 - q_1)/(p_2 - p_1) > 0$, so with $-\mu_{A_2} > 0$,

$$\mu_{A_1} = R_Q(-\mu_{A_2}) > 0,$$

and μ is sign-compatible with S in (S14); that is, μ is a signature and the network admits multiple steady states.

Consistency with the injectivity test. The Maple computation of $\det(M^*)$ for the OF network is

$$\begin{aligned}
\det := & -p_1p_3p_4q_5\,k_1k_2k_3k_4k_5\,z_1z_5z_7z_8z_9 + p_2p_3p_4q_5\,k_1k_2k_3k_4k_5\,z_2z_5z_7z_8z_9 \\
& - p_1p_3p_4\,k_1k_2k_3k_4k_5\,z_1z_5z_6z_7z_8 + p_1p_3q_5\,k_1k_2k_3k_4k_5\,z_1z_4z_5z_7z_9 \\
& - p_1p_4q_5\,k_1k_2k_3k_4k_5\,z_1z_3z_5z_8z_9 + p_1q_3q_5\,k_1k_2k_3k_4k_5\,z_1z_3z_5z_7z_9 \\
& + p_2p_3p_4\,k_1k_2k_3k_4k_5\,z_2z_5z_6z_7z_8 - p_2p_3q_5\,k_1k_2k_3k_4k_5\,z_2z_4z_5z_7z_9 \\
& + p_2p_4q_5\,k_1k_2k_3k_4k_5\,z_2z_3z_5z_8z_9 - p_2q_3q_5\,k_1k_2k_3k_4k_5\,z_2z_3z_5z_7z_9 \\
& + p_1p_3\,k_1k_2k_3k_4k_5\,z_1z_4z_5z_6z_7 - 2p_1p_4\,k_1k_2k_3k_4k_5\,z_1z_3z_5z_6z_8 \\
& + p_1p_5\,k_1k_2k_3k_4k_5\,z_1z_3z_5z_6z_9 + p_1q_3\,k_1k_2k_3k_4k_5\,z_1z_3z_5z_6z_7 \\
& - p_2p_3\,k_1k_2k_3k_4k_5\,z_2z_4z_5z_6z_7 + 2p_2p_4\,k_1k_2k_3k_4k_5\,z_2z_3z_5z_6z_8 \\
& - p_2p_5\,k_1k_2k_3k_4k_5\,z_2z_3z_5z_6z_9 - p_2q_3\,k_1k_2k_3k_4k_5\,z_2z_3z_5z_6z_7.
\end{aligned} \tag{S15}$$

Under the positive-class conditions, augmented by $p_4 > 0$, $p_5 > 0$, $q_5 > 0$, the two terms

$$-2p_1p_4\,k_1k_2k_3k_4k_5\,z_1z_3z_5z_6z_8 \qquad \text{and} \qquad +2p_2p_4\,k_1k_2k_3k_4k_5\,z_2z_3z_5z_6z_8$$

have opposite signs (the first negative since $p_1, p_4 > 0$, the second positive since $p_2, p_4 > 0$). Hence $\det(M^*)$ is not sign-definite and the network is non-injective. Non-injectivity is consistent with the MSA conclusion that the positive-class OF system is multistationary. □

S6.3 Proof of part (ii): the negative class is monostationary

Take μ as in (S13) with $\mu_{A_3} < \mu_{A_2} < M_1 < 0$, $\mu_{A_4} = \mu_{A_{12}} = M_2 < 0$, $M_3 < 0$, $p_1 > 0 > p_2$, $q_3 > 0$. As in part (i), $q_3 > 0$, $p_3 > 0$, and $p_3\mu_{A_2} > M_3$ give $\mu_{A_3} < 0$, and the master criterion requires $\mu_{A_1} > 0$ for a signature.

Under $q_2 > q_1 > 0$ and $p_1 > 0 > p_2$ the ratio

$$R_Q = \frac{q_2 - q_1}{p_2 - p_1} < 0,$$

so with $-\mu_{A_2} > 0$,

$$\mu_{A_1} = R_Q(-\mu_{A_2}) < 0.$$

Since $\mu_{A_1} \not> 0$, μ is not sign-compatible with S and is not a signature; the same conclusion holds for the remaining cases of Step 13. Hence the network has no capacity for multiple steady states.

For the injectivity test, under $q_2 > q_1 > 0$, $p_1 > 0 > p_2$, $q_3 > 0$, $p_3 > 0$ together with $p_4 < 0$, $p_5 > 0$, $q_5 > 0$, every term of $\det(M^*)$ in (S15) carries a positive coefficient, so the network is injective. Injectivity confirms monostationarity, in agreement with the MSA. □

S6.4 Proof of part (iii): the Q-null class is monostationary

Take μ as in (S13) with the same shelf conditions as in part (ii), now with $q_2 = q_1$. As before, $q_3 > 0$, $p_3 > 0$, $p_3\mu_{A_2} > M_3$ give $\mu_{A_3} < 0$, and a signature requires $\mu_{A_1} > 0$. With $q_2 = q_1$,

$$R_Q = \frac{q_2 - q_1}{p_2 - p_1} = 0 \quad \Longrightarrow \quad \mu_{A_1} = R_Q(-\mu_{A_2}) = 0.$$

Since $\mu_{A_1} \not> 0$, μ is not a signature, and the remaining Step 13 cases give the same result. The network is therefore monostationary. Consistently, under $q_2 = q_1$, $p_1 > 0 > p_2$, $q_3 > 0$, $p_3 > 0$, $p_4 < 0$, $p_5 > 0$, $q_5 > 0$ all terms of $\det(M^*)$ in (S15) are positive, so the network is injective. □

S6.5 Proof of part (iv): the P-null class is monostationary

For the P-null class it is convenient to write the master solution with μ_{A_1} free:

$$\mu = \begin{bmatrix} \mu_{A_1} \\ \dfrac{p_2 - p_1}{q_2 - q_1}(-\mu_{A_1}) \\ \dfrac{M_3 - \frac{p_2-p_1}{q_2-q_1} p_3(-\mu_{A_1})}{q_3} \\ M_2 \\ M_2 \end{bmatrix}, \qquad M_3 < 0,\ q_2 \neq q_1,\ q_3 > 0. \tag{S16}$$

Suppose $p_2 = p_1$, so $R_P = (p_2 - p_1)/(q_2 - q_1) = 0$. Then

$$\mu_{A_2} = R_P(-\mu_{A_1}) = 0,$$

contradicting the shelving requirement $\mu_{A_2} < 0$. The system is inconsistent, μ is not a signature, and the same holds for the other Step 13 cases; hence the network has no capacity for multiple steady states.

For the injectivity test, recall that $\det(M^*)$ in (S15) can be made all-positive only when $p_1 > 0 > p_2$ and all-negative only when $p_1 < 0 < p_2$. With $p_1 = p_2$ neither is possible, so $\det(M^*)$ is not sign-definite and the network is non-injective. Since non-injectivity is only a necessary condition for multistationarity, a non-injective network may still be monostationary; here the MSA shows that this is exactly what happens, for all Step 13 cases. □

S7 Proof of Proposition 3 (Soil Carbon Sequestration)

The SCS network has deficiency 1 and is handled by the DOA. Throughout, the emission kinetic orders satisfy $e_{4,13} \geq 1$, $f_{4,13} \leq 0$, and $e_{4,13} + f_{4,13} = 1$; these are the quantities written e_{13}, f_{13} in the proposition statement.

S7.1 Setup for the SCS network

Confluence vector. Solving the confluence conditions on the complex set $\{A_1 + 2A_2, 2A_1 + A_2, A_1, A_2, A_3, A_4, A_{13}\}$ yields the general form $h = [-g, g, -g, g, 0, 0, 0]^\mathsf{T}$ for $g \in \mathbb{R}_{>0}$; taking $g = 1$,

$$h = [h_{A_1+2A_2}, h_{2A_1+A_2}, h_{A_1}, h_{A_2}, h_{A_3}, h_{A_4}, h_{A_{13}}]^\mathsf{T} = [-1, 1, -1, 1, 0, 0, 0]^\mathsf{T}. \tag{S17}$$

***T*-matrix.** With species ordered $A_1, A_2, A_3, A_4, A_{13}$,

$$T = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_{13} \end{array} \begin{array}{c} \begin{array}{ccccccc} A_1{+}2A_2 & 2A_1{+}A_2 & A_2 & A_3 & A_4 & A_{13} & A_1 \end{array} \\ \begin{bmatrix} p_1 & p_2 & 0 & 0 & 0 & 0 & 1 \\ q_1 & q_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{4,13} & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{4,13} & 1 & 0 \end{bmatrix} \end{array}. \tag{S18}$$

UML partitions. Writing $\mathscr{C}_1 = A_1 + 2A_2$, $\mathscr{C}_2 = 2A_1 + A_2$, $\mathscr{C}_3 = A_2$, $\mathscr{C}_4 = A_3$, $\mathscr{C}_5 = A_4$, $\mathscr{C}_6 = A_{13}$, $\mathscr{C}_7 = A_1$, the admissible partitions are listed in Table S3. The proofs use Partitions A, C, and G as indicated.

Partition	Upper	Middle	Lower
A	$\{\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3,\mathscr{C}_4\}$	$\{\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\}$
B	$\{\}$	$\{\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3,\mathscr{C}_4\}$
C	$\{\mathscr{C}_1,\mathscr{C}_2\}$	$\{\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\mathscr{C}_3,\mathscr{C}_4\}$
D	$\{\mathscr{C}_3,\mathscr{C}_4\}$	$\{\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\mathscr{C}_1,\mathscr{C}_2\}$
E	$\{\mathscr{C}_1,\mathscr{C}_2\}$	$\{\mathscr{C}_3,\mathscr{C}_4,\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\}$
F	$\{\}$	$\{\mathscr{C}_3,\mathscr{C}_4,\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\mathscr{C}_1,\mathscr{C}_2\}$
G	$\{\mathscr{C}_3,\mathscr{C}_4\}$	$\{\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\}$
H	$\{\}$	$\{\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\mathscr{C}_3,\mathscr{C}_4\}$
I	$\{\}$	$\{\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3,\mathscr{C}_4,\mathscr{C}_5,\mathscr{C}_6,\mathscr{C}_7\}$	$\{\}$

Table S3. Upper–middle–lower partitions of the reactant complexes of the SCS network.

Stoichiometric subspace. For all parts,

$$S = \left\{ \begin{bmatrix} a_1 \\ -a_1 - a_2 + a_3 \\ a_2 \\ -a_3 + a_4 \\ -a_4 \end{bmatrix} \middle| \; a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}. \tag{S19}$$

The candidate displacement is $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)^{\mathsf{T}}$, and using $e_{4,13} + f_{4,13} = 1$ the middle-part equalities reduce to $\mu_1 = \mu_4 = \mu_{13}$.

Proposition 3 (Soil Carbon Sequestration)**.** *The following hold for the SCS system:*

(i) ***Positive class.*** *If $R = (p_2 - p_1)/(q_2 - q_1) > 0$ (e.g. $p_2 > p_1 > 0$ and $q_2 > q_1 > 0$, or $p_1 > p_2 > 0$ and $q_1 > q_2 > 0$), then the system is multistationary.*

(ii) ***Negative class.*** *If $p_1, q_2 < 0$ and $p_2, q_1 > 0$, together with $e_{13} \geq 1$ and $f_{13} \leq 0$, then the system is monostationary.*

(iii) ***Q-null class.*** *If $q_2 = q_1$ and $p_1 > p_2 > 0$, then the system is multistationary.*

(iv) ***P-null class.*** *If $p_2 = p_1$ and $q_1 \neq q_2$ (with $q_1 \neq 0$), then the system is multistationary.*

S7.2 Proof of the positive-class case

Use Partition A: $U = \{A_1 + 2A_2, 2A_1 + A_2, A_2, A_3\}$, $M = \{A_4, A_{13}, A_1\}$, $L = \{\}$. The DOA yields $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)^{\mathsf{T}}$ (the equalities of the middle part give $\mu_1 = \mu_4 = \mu_{13}$), subject to the ordering $\mu_2 \geq \mu_3 > \mu_1$ together with the cut-pair and between-part inequalities

$$\begin{aligned} &\text{(I)} \quad p_2\mu_1 + q_2\mu_2 \; > \; p_1\mu_1 + q_1\mu_2, \\ &\text{(II)} \quad p_1\mu_1 + q_1\mu_2 \; > \; \mu_1. \end{aligned} \tag{S20}$$

Rearranged, these read

$$\text{(I) } (p_2 - p_1)\mu_1 + (q_2 - q_1)\mu_2 > 0, \qquad \text{(II) } (p_1 - 1)\mu_1 + q_1\mu_2 > 0. \tag{S21}$$

Recall the stoichiometric subspace (S19), whose generic element is $\sigma = (a_1, -a_1 - a_2 + a_3, a_2, -a_3 + a_4, -a_4)$; sign-compatibility of $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)$ with σ requires

$$\operatorname{sign}(\mu_1) = \operatorname{sign}(a_1) = \operatorname{sign}(-a_4) = \operatorname{sign}(a_4 - a_3), \quad \operatorname{sign}(\mu_2) = \operatorname{sign}(-a_1 - a_2 + a_3), \quad \operatorname{sign}(\mu_3) = \operatorname{sign}(a_2). \tag{S22}$$

We exhibit a signature with $\mu_1 < 0$ and $\mu_2 > 0$. Take the positive-class branch $p_2 > p_1 > 0$ and $q_2 > q_1 > 0$ (so $R = (p_2 - p_1)/(q_2 - q_1) > 0$); the reversed branch $p_1 > p_2 > 0$, $q_1 > q_2 > 0$ is symmetric. Fix any $\mu_1 < 0$. In (S21) the coefficients of μ_2, namely $q_2 - q_1 > 0$ in (I) and $q_1 > 0$ in (II), are positive, so choosing

$$\mu_2 \; > \; \max\left\{ \frac{-(p_2 - p_1)\mu_1}{q_2 - q_1}, \; \frac{-(p_1 - 1)\mu_1}{q_1} \right\} \; > \; 0 \tag{S23}$$

makes both (I) and (II) strict, and $\mu_2 > 0 > \mu_1$ is consistent with the ordering $\mu_2 \geq \mu_3 > \mu_1$ for any $\mu_3 \in (\mu_1, \mu_2]$. Thus μ has sign pattern $(\operatorname{sign}\mu_1, \operatorname{sign}\mu_2) = (-, +)$.

It remains to realize (S22). With $\mu_1 < 0$ we need $a_1 < 0$, $a_4 > 0$, and $a_3 > a_4 > 0$ (so $a_4 - a_3 < 0$); then $-a_1 + a_3 > 0$, and with $\mu_2 > 0$ we need $-a_1 - a_2 + a_3 > 0$, i.e. $a_2 < -a_1 + a_3$, which holds for a suitable a_2. Choosing μ_3 to match $\mathrm{sign}(a_2)$ (any of $a_2 < 0$, $a_2 = 0$, or $0 < a_2 < -a_1 + a_3$ is admissible, giving $\mu_3 < 0$, $\mu_3 = 0$, or $\mu_3 > 0$ respectively, all consistent with $\mu_2 \geq \mu_3 > \mu_1$) completes a sign-compatible σ. For instance, with $p_1, p_2, q_1, q_2 = (1,2,1,2)$ and $\mu_1 = -1$, $\mu_2 = 2$, $\mu_3 = -\frac{1}{2}$, the vector $\sigma = (-2,6,-2,-1,-1)$ from $(a_1,a_2,a_3,a_4) = (-2,-2,2,1)$ is sign-compatible with μ, and (I)–(II) hold ($2 > 1 > -1$). Hence a nonzero $\mu \in \mathbb{R}^{\mathscr{S}}$ sign-compatible with S exists, and the positive-class SCS system is multistationary. This agrees with the injectivity test: under the positive-class sign conditions, $\det(M^*)$ in (S24) contains terms of both signs (e.g. $-e_{4,13}p_1k_1k_2k_4k_5z_1z_3z_5z_6 < 0$ and $+e_{4,13}p_2k_1k_2k_4k_5z_2z_3z_5z_6 > 0$), so the network is non-injective. ☐

S7.3 Proof of the negative-class case

Assume the negative class $R = (p_2 - p_1)/(q_2 - q_1) < 0$, realized by $p_1, q_2 < 0$ and $p_2, q_1 > 0$. The Maple script of [1] gives

$$
\begin{aligned}
\det(M^*) = &-e_{4,13}p_1k_1k_2k_4k_5z_1z_3z_5z_6 - e_{4,13}p_1k_1k_3k_4k_5z_1z_4z_5z_6 \\
&+ e_{4,13}p_2k_1k_2k_4k_5z_2z_3z_5z_6 + e_{4,13}p_2k_1k_3k_4k_5z_2z_4z_5z_6 \\
&+ e_{4,13}q_1k_1k_2k_3k_4z_1z_4z_5z_7 + e_{4,13}q_1k_2k_3k_4k_5z_1z_4z_5z_6 \\
&- e_{4,13}q_2k_1k_2k_3k_4z_2z_4z_5z_7 - e_{4,13}q_2k_2k_3k_4k_5z_2z_4z_5z_6 \\
&- f_{4,13}q_1k_1k_2k_3k_5z_1z_4z_5z_7 + f_{4,13}q_2k_1k_2k_3k_5z_2z_4z_5z_7 \\
&+ e_{4,13}k_1k_2k_4k_5z_3z_5z_6z_7 + e_{4,13}k_1k_3k_4k_5z_4z_5z_6z_7 \\
&+ q_1k_1k_2k_3k_5z_1z_4z_6z_7 - q_2k_1k_2k_3k_5z_2z_4z_6z_7.
\end{aligned}
\tag{S24}
$$

With $e_{4,13} \geq 1$ and $f_{4,13} \leq 0$, the conditions $p_1, q_2 < 0$ and $p_2, q_1 > 0$ make every term of (S24) positive. Hence the SCS network is injective and therefore monostationary; each stoichiometric compatibility class contains at most one steady state. ☐

S7.4 Proof of the Q-null and P-null cases

Q-null class ($q_2 = q_1$, $p_1 > p_2 > 0$). Use Partition A again, with the same displacement form $\mu = (\mu_1, \mu_2, \mu_3, \mu_1, \mu_1)^{\mathsf{T}}$ and the inequalities (S21). With $q_2 = q_1$, inequality (I) becomes $(p_2 - p_1)\mu_1 > 0$; since $p_1 > p_2$ gives $p_2 - p_1 < 0$, this holds precisely when $\mu_1 < 0$. Fix $\mu_1 < 0$. Inequality (II), $(p_1 - 1)\mu_1 + q_1\mu_2 > 0$ with $q_1 > 0$, is then satisfied by choosing

$$\mu_2 > \frac{-(p_1 - 1)\mu_1}{q_1} > 0, \tag{S25}$$

which is compatible with the ordering $\mu_2 \geq \mu_3 > \mu_1$. The resulting sign pattern is again $(\mathrm{sign}\,\mu_1, \mathrm{sign}\,\mu_2) = (-,+)$, identical to the positive-class case, so by the sign-compatibility analysis (S22) a nonzero μ sign-compatible with S exists (e.g. $\sigma = (-2,6,-2,-1,-1)$ from $(a_1,a_2,a_3,a_4) = (-2,-2,2,1)$, with $\mu_3 < 0$). Hence the Q-null SCS system is multistationary.

P-null class ($p_2 = p_1$, $q_1 \neq q_2$, $q_1 \neq 0$). Use Partition A once more. With $p_2 = p_1$, inequality (I) becomes $(q_2 - q_1)\mu_2 > 0$ and inequality (II) becomes $(p_1 - 1)\mu_1 + q_1\mu_2 > 0$. Take the branch $q_2 > q_1 > 0$; then (I) holds iff $\mu_2 > 0$, and fixing $\mu_1 < 0$, (II) is satisfied by choosing μ_2 as in (S23) (the threshold from (I) is now 0, so only the (II) threshold binds). This gives the sign pattern $(-,+)$ and, by (S22), a sign-compatible signature (again $\sigma = (-2,6,-2,-1,-1)$ works). The branch $q_1 > q_2 > 0$ is symmetric, exchanging the roles of μ_1 and μ_2 via Partition C and giving sign pattern $(+,-)$. In either branch a nonzero μ sign-compatible with S exists, so the P-null SCS system is multistationary. ☐

S8 Proof of Proposition 4 (Wetland Restoration)

The WR network has deficiency 2 and is handled by the MSA. The defining structural feature is the bidirectional atmosphere–storage coupling carried by the kinetic orders q_4, p_4, q_5, p_5, which generates the auxiliary ratio governing the Q-null class.

S8.1 Setup for the WR network

Orientation and kernel. Choose $\mathscr{O} = \{R_1, R_3, R_5, R_{14,6}, R_{14,8}, R_{14,9}\}$, with map and kernel

$$L_{\mathscr{O}} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_{14} \end{array}, \qquad \ker L_{\mathscr{O}} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \begin{array}{l} R_1 \\ R_3 \\ R_5 \\ R_{14,6} \\ R_{14,8} \\ R_{14,9} \end{array}.$$

Classes, representatives, shelving. The equivalence and fundamental classes are

$$\begin{aligned} \mathscr{P}_0 &= \{R_3\}, & \mathscr{C}_0 &= \{R_3, R_4\}; \\ \mathscr{P}_1 &= \{R_1, R_{14,9}\}, & \mathscr{C}_1 &= \{R_1, R_2, R_{14,9}, R_{14,10}\}; \\ \mathscr{P}_2 &= \{R_5, R_{14,6}\}, & \mathscr{C}_2 &= \{R_5, R_{14,6}\}; \\ \mathscr{P}_3 &= \{R_{14,8}\}, & \mathscr{C}_3 &= \{R_{14,8}, R_{14,7}\}. \end{aligned}$$

Picking $W = \{R_1, R_5, R_{14,7}\}$ gives $\ker^{\perp} L_{\mathscr{O}} \cap \Gamma_W = \operatorname{span}\{(-1,1,1)^{\mathsf{T}}\}$, whose basis graph is a forest, so the system is linear. The shelving is $\mathscr{L}_1 = \{R_1, R_2, R_{14,9}, R_{14,10}\}$; $\mathscr{M}_2 = \{R_5, R_{14,6}\}$; $\mathscr{L}_3 = \{R_{14,7}, R_{14,8}\}$, all others empty.

Linear system and master solution. With $M_1, M_2, M_3 < 0$ large and negative, the MSA yields

$$\begin{aligned} \mu_{A_{14}} < \mu_{A_1} < \frac{q_1 - q_2}{p_2 - p_1}\, \mu_{A_2} &< M_1, \\ e_{14}\mu_{A_4} + f_{14}\mu_{A_{14}} = \mu_{A_{14}} &= M_2, \\ \mu_{A_{14}} < \frac{q_4 - q_5}{p_5 - p_4}\, \mu_{A_2} &< M_3, \\ \mu_{A_2} &= \mu_{A_3}. \end{aligned} \tag{S26}$$

Since $e_{14} + f_{14} = 1$ (with $e_{14} \geq 1$, $f_{14} \leq 0$), the second line gives $\mu_{A_4} = \mu_{A_{14}} = M_2 < 0$. The first and third lines give $\mu_{A_{14}} < \mu_{A_1} < 0$ and

$$\underbrace{\frac{(q_1 - q_2)(p_5 - p_4) + (q_4 - q_5)(p_2 - p_1)}{(p_2 - p_1)(p_5 - p_4)}}_{=: R_{WR}}\, \mu_{A_2} < M_1 + M_3 < 0. \tag{S27}$$

Together with $\mu_{A_2} = \mu_{A_3}$, the signature is

$$\mu = \left[\mu_{A_1},\ \mu_{A_2},\ \mu_{A_2},\ \mu_{A_{14}},\ \mu_{A_{14}}\right]^{\mathsf{T}}. \tag{S28}$$

The stoichiometric subspace is

$$S = \left\{ \begin{bmatrix} a \\ -a - b + c + d \\ b \\ -c \\ -d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}, \tag{S29}$$

with generic element α. Since $\mu_{A_1}, \mu_{A_{14}} < 0$, their signs are negative; the sign of μ_{A_2} is determined by R_{WR} via (S27): $R_{WR} > 0$ gives $\mu_{A_2} < 0$, while $R_{WR} < 0$ gives $\mu_{A_2} > 0$. (The value $R_{WR} = 0$ is excluded, since it would give $0 < M_1 + M_3 < 0$.) Whether μ is a signature thus reduces to the sign of R_{WR}.

Proposition 4 (Wetland Restoration)**.** *The following hold for the WR system:*

(i) ***Positive class.*** *If $p_1 \neq p_2$ or $p_4 \neq p_5$, together with $e_{14} > 0$ and $f_{14} < 0$, then the system is monostationary.*

(ii) ***Negative class.*** *If $p_1 \neq p_2$ or $p_4 \neq p_5$, then the system is multistationary.*

(iii) ***Q-null class.***

- *(a) If $(q_4 - q_5)/(p_5 - p_4) > 0$ and $p_4 \neq p_5$, then the system is monostationary.*
- *(b) If $(q_4 - q_5)/(p_5 - p_4) < 0$ and $p_4 \neq p_5$, then the system is multistationary.*

S8.2 Proof of part (i): the positive class is monostationary

Take μ as in (S28) with $\mu_{A_{14}} < \mu_{A_1} < 0$ and $R_{WR}\mu_{A_2} < 0$. In the positive class $R_{WR} > 0$, so $\mu_{A_2} < 0$ and every entry of μ is negative: $\mathrm{sign}(\mu) = (-,-,-,-,-)$. Sign-compatibility with $\alpha \in S$ in (S29) would require

$$a < 0\ (\Rightarrow -a > 0), \quad b < 0\ (\Rightarrow -b > 0), \quad -c < 0\ (\Rightarrow c > 0), \quad -d < 0\ (\Rightarrow d > 0),$$

so $-a-b+c+d > 0$. But the second entry of α must be negative (matching $\mu_{A_2} < 0$), a contradiction. Hence no signature exists and the positive-class WR system is monostationary.

Consistency with the injectivity test. The Maple script of [1] gives the determinant of M^* for the WR network:

$$\begin{aligned}
\det = {} & e_{14}p_1q_4k_1k_2k_3k_4z_1z_4z_5z_7 - e_{14}p_1q_5k_1k_2k_3k_4z_1z_4z_5z_8 \\
& - e_{14}p_1p_4k_1k_2k_4k_5z_1z_3z_5z_7 - e_{14}q_1p_4k_1k_3k_4k_5z_1z_4z_5z_7 \\
& + e_{14}p_1p_5k_1k_2k_4k_5z_1z_3z_5z_8 + e_{14}p_1p_5k_1k_3k_4k_5z_1z_4z_5z_8 \\
& - e_{14}p_2q_4k_1k_2k_3k_4z_2z_4z_5z_7 + e_{14}p_2q_5k_1k_2k_3k_4z_2z_4z_5z_8 \\
& + e_{14}p_2p_4k_1k_2k_4k_5z_2z_3z_5z_7 + e_{14}p_2p_4k_1k_3k_4k_5z_2z_4z_5z_7 \\
& - e_{14}p_2p_5k_1k_2k_4k_5z_2z_3z_5z_8 - e_{14}p_2p_5k_1k_3k_4k_5z_2z_4z_5z_8 \\
& + e_{14}q_1p_4k_2k_3k_4k_5z_1z_4z_5z_7 - e_{14}q_1p_5k_2k_3k_4k_5z_1z_4z_5z_8 \\
& - e_{14}q_2p_4k_2k_3k_4k_5z_2z_4z_5z_7 + e_{14}q_2p_5k_2k_3k_4k_5z_2z_4z_5z_8 \\
& - f_{14}p_1q_4k_1k_2k_3k_5z_1z_4z_5z_7 + f_{14}p_1q_5k_1k_2k_3k_5z_1z_4z_5z_8 \\
& + f_{14}p_2q_4k_1k_2k_3k_5z_2z_4z_5z_7 - f_{14}p_2q_5k_1k_2k_3k_5z_2z_4z_5z_8 \\
& - e_{14}p_1k_1k_2k_4k_5z_1z_3z_5z_6 - e_{14}p_1k_1k_2k_4k_5z_1z_3z_5z_{10} \\
& - e_{14}p_1k_1k_3k_4k_5z_1z_4z_5z_6 - e_{14}p_1k_1k_3k_4k_5z_1z_4z_5z_{10} \\
& + e_{14}p_2k_1k_2k_4k_5z_2z_3z_5z_6 + e_{14}p_2k_1k_2k_4k_5z_2z_3z_5z_{10} \\
& + e_{14}p_2k_1k_3k_4k_5z_2z_4z_5z_6 + e_{14}p_2k_1k_3k_4k_5z_2z_4z_5z_{10} \\
& - e_{14}q_4k_1k_2k_3k_4z_4z_5z_7z_9 - e_{14}q_4k_2k_3k_4k_5z_4z_5z_7z_{10} \\
& + e_{14}q_5k_1k_2k_3k_4z_4z_5z_8z_9 + e_{14}q_5k_2k_3k_4k_5z_4z_5z_8z_{10} \\
& + e_{14}q_1k_1k_2k_3k_4z_1z_4z_5z_9 + e_{14}q_1k_2k_3k_4k_5z_1z_4z_5z_6 \\
& + e_{14}q_1k_2k_3k_4k_5z_1z_4z_5z_{10} - e_{14}q_2k_1k_2k_3k_4z_2z_4z_5z_9 \\
& - e_{14}q_2k_2k_3k_4k_5z_2z_4z_5z_6 - e_{14}q_2k_2k_3k_4k_5z_2z_4z_5z_{10} \\
& + e_{14}p_4k_1k_2k_4k_5z_3z_5z_7z_9 + e_{14}p_4k_1k_3k_4k_5z_4z_5z_7z_9 \\
& - e_{14}p_5k_1k_2k_4k_5z_3z_5z_8z_9 - e_{14}p_5k_1k_3k_4k_5z_4z_5z_8z_9 \\
& + f_{14}q_4k_1k_2k_3k_5z_4z_5z_7z_9 - f_{14}q_5k_1k_2k_3k_5z_4z_5z_8z_9 \\
& - f_{14}q_1k_1k_2k_3k_5z_1z_4z_5z_9 + f_{14}q_2k_1k_2k_3k_5z_2z_4z_5z_9 \\
& + p_1q_4k_1k_2k_3k_5z_1z_4z_6z_7 - p_1q_5k_1k_2k_3k_5z_1z_4z_6z_8 \\
& - p_2q_4k_1k_2k_3k_5z_2z_4z_6z_7 + p_2q_5k_1k_2k_3k_5z_2z_4z_6z_8 \\
& + e_{14}k_1k_2k_4k_5z_3z_5z_6z_9 + e_{14}k_1k_3k_4k_5z_4z_5z_6z_9 \\
& - q_4k_1k_2k_3k_5z_4z_6z_7z_9 + q_5k_1k_2k_3k_5z_4z_6z_8z_9 \\
& + q_1k_1k_2k_3k_5z_1z_4z_6z_9 - q_2k_1k_2k_3k_5z_2z_4z_6z_9.
\end{aligned} \tag{S30}$$

Isolating the terms

$$\begin{gathered}
e_{14}k_1k_3k_4k_5z_4z_5z_6z_9, \quad e_{14}k_1k_2k_4k_5z_3z_5z_6z_9, \quad -q_4k_1k_2k_3k_5z_4z_6z_7z_9, \\
q_5k_1k_2k_3k_5z_4z_6z_8z_9, \quad q_1k_1k_2k_3k_5z_1z_4z_6z_9, \quad -q_2k_1k_2k_3k_5z_2z_4z_6z_9,
\end{gathered}$$

the assignment $e_{14} > 0$, $q_4 < 0$, $q_5 > 0$, $q_1 > 0$, $q_2 < 0$, $p_1 < 0$, $p_2 > 0$, $p_4 > 0$, $p_5 < 0$, $f_{14} < 0$ makes every term of $\det(M^*)$ in (S30) positive; this is the only sign assignment that renders all terms positive. Since $e_{14} \not< 0$, no all-negative assignment is possible. Hence the positive-class WR system is injective, and by algebraic rearrangement this is exactly the regime $R_{WR} > 0$, in agreement with the MSA conclusion of monostationarity. $\square$

S8.3 Proof of part (ii): the negative class is multistationary

Take μ as in (S28) with $\mu_{A_{14}} < \mu_{A_1} < 0$ and $R_{WR}\mu_{A_2} < 0$, and assume $R_{WR} < 0$, so $\mu_{A_2} > 0$. Then $\text{sign}(\mu) = (-,+,+,-,-)$, and matching against $\alpha \in S$ gives

$$a < 0\ (\Rightarrow -a > 0), \quad b > 0, \quad -c < 0\ (\Rightarrow c > 0), \quad -d < 0\ (\Rightarrow d > 0),$$

and $-a-b+c+d > 0$, i.e. $-a+c+d > b$. Since $-a > 0$, $c > 0$, $d > 0$ we have $-a+c+d > 0$, and with $b > 0$ the inequality $-a+c+d > b > 0$ holds. Thus the signs of μ are sign-compatible with those of α, a signature exists, and the negative-class WR system is multistationary. □

S8.4 Proof of part (iii): the Q-null cases and the role of $(q_4 - q_5)/(p_5 - p_4)$

In the Q-null class $q_1 = q_2$, so the numerator $(q_1 - q_2)(p_5 - p_4)$ of R_{WR} in (S27) vanishes and

$$R_{WR} = \frac{q_4 - q_5}{p_5 - p_4}.$$

The sign of μ_{A_2}, and hence the steady-state capacity of the network, is therefore governed entirely by the auxiliary ratio $(q_4 - q_5)/(p_5 - p_4)$ introduced by the bidirectional atmosphere–storage coupling.

(a) $(q_4 - q_5)/(p_5 - p_4) > 0$ **with** $p_4 \neq p_5$**.** Then $R_{WR} > 0$, so $\mu_{A_2} < 0$ and $\text{sign}(\mu) = (-,-,-,-,-)$. By the argument of part (i), this is not sign-compatible with any $\alpha \in S$; no signature exists and the Q-null WR system is monostationary.

(b) $(q_4 - q_5)/(p_5 - p_4) < 0$ **with** $p_4 \neq p_5$**.** Then $R_{WR} < 0$, so $\mu_{A_2} > 0$ and $\text{sign}(\mu) = (-,+,+,-,-)$. By the argument of part (ii), this is sign-compatible with some $\alpha \in S$; a signature exists and the Q-null WR system is multistationary.

In summary, in the Q-null class the steady-state capacity is decided by the sign of μ_{A_2}, equivalently by $R_{WR} = (q_4 - q_5)/(p_5 - p_4)$: the case $\mu_{A_2} > 0$ (i.e. $R_{WR} < 0$) yields multistationarity, while $\mu_{A_2} < 0$ (i.e. $R_{WR} > 0$) yields monostationarity. □

Note on notation

Two notational conventions are used consistently across the proofs. First, the emission kinetic orders written e_{13}, f_{13} in the SCS proposition statement are the same quantities written $e_{4,13}, f_{4,13}$ in the SCS derivation (and analogously e_{11}, f_{11} for BCS, e_{14}, f_{14} for WR); all satisfy $e_i \geq 1$, $f_i \leq 0$, $e_i + f_i = 1$. Second, the Anderies ratio of the main text is $R = (p_2 - p_1)/(q_2 - q_1)$; inside the DOA displays we write $R_p = (p_2 - p_1)/(q_2 - q_1) = R$ (used when $q_2 \neq q_1$) and $R_q = (q_2 - q_1)/(p_2 - p_1) = 1/R$ (used when $p_2 \neq p_1$), so that the P-null condition $p_2 = p_1$ corresponds to $R_p = 0$ and the Q-null condition $q_2 = q_1$ corresponds to $R_q = 0$.

S9 Sensitivity to the Storage Parameterization

The storage parameters enter the analysis only through the emission kinetic orders. Writing $c_i = (1 - \lambda_i) + \mu_i \lambda_i$ for the fraction of CDR storage withheld from emission, the logarithmic linearization of Eq. (5) of the main text about a positive steady state (A_4^*, A_i^*) gives

$$e_i \ = \ \frac{A_4^*}{A_4^* - c_i A_i^*}, \qquad f_i \ = \ \frac{-c_i A_i^*}{A_4^* - c_i A_i^*}, \tag{S31}$$

so that $e_i + f_i = 1$ identically, with $e_i \geq 1$ and $f_i \leq 0$ whenever $c_i A_i^* < A_4^*$. Two points follow that are not evident from Table 3 of the main text alone. First, λ_i and μ_i act only through the single combination c_i. Second, the kinetic orders depend additionally on the operating point A_i^*/A_4^* at which the linearization is taken, so they are not fixed by Table 3 of the main text by itself.

For the four systems analyzed here, Table 3 of the main text gives $c_{11} = 0.5$ (BCS), $c_{12} = 0$ (OF), $c_{13} = 0.99$ (SCS) and $c_{14} = 0.999$ (WR). Two consequences bear directly on Section 4 of the main text. Ocean fertilization is the only system in the set with $c_i = 0$; this is what forces $e_{12} = 1$, $f_{12} = 0$ and reduces the emission reaction to linear mass action, and any leak at all would move it off that form. Wetland restoration sits at the opposite extreme, and Proposition 4(i) requires $f_{14} < 0$ strictly, a condition met only because $\lambda_{14} = 0.01$ and $\mu_{14} = 0.9$ give $c_{14} \neq 0$. Under a parameterization with $\lambda_{14} = 1$ and $\mu_{14} = 0$ one would have $f_{14} = 0$ and Proposition 4(i) would not apply.

Conditions that require only $e_i \geq 1$ and $f_i \leq 0$, namely Propositions 1(ii) and 3(ii), hold for every admissible pair (λ_i, μ_i) and are robust to recalibration. Conditions that turn on a strict inequality or on the exact value $c_i = 0$, namely Proposition 2 and Proposition 4(i), are not. Recalibration of Table 3 of the main text against measured data could therefore change which propositions apply to a given technology, even though the network structure is unchanged.

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